The Effect of Graph Connectivity on Metastability in a Stochastic System of Spiking Neurons

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Abstract

We consider a continuous-time stochastic model of spiking neurons originally introduced by Ferrari et al. in [8]. In this model, we have a finite or countable number of neurons which are vertices in some graph G where the edges indicate the synaptic connection between them. We focus on metastability, understood as the property for the time of extinction of the network to be asymptotically memory-less, and we prove that this model exhibits two different behaviors depending on the nature of the specific underlying graph of interaction G that is chosen. In this model the spiking activity of any given neuron is represented by a point process, which rate fluctuate between 1 and 0 over time depending on whether the membrane potential is positive or null. The membrane potential of each neuron evolves in time by integrating all the spikes of its *pre-synaptic* neurons up to the last spike of the said neuron, so that when a neuron spikes, its membrane potential is reset to 0 while the membrane potential of each if its *post-synaptic* neurons is increased by one unit. Moreover each neuron is exposed to a leakage effect, modeled as an abrupt loss of membrane potential which occurs at random times driven by a Poisson process of some fixed rate γ. It was previously proven that when the graph G is the infinite one-dimensional lattice, this model presents a phase transition with respect to the parameter γ. It was also proven that, when γ is small enough, the renormalized time of extinction (the first time at which all neurons have a null membrane potential) of a finite version of the system converges in law toward an exponential random variable when the number of neurons goes to infinity. The present article is divided into two parts. First we prove that, in the finite one-dimensional lattice, this last result doesn't hold anymore if γ is not small anymore, in fact we prove that for $\gamma > 1$ the renormalized time of extinction is asymptotically deterministic. Then we prove that conversely, if G is the complete graph, the result of metastability holds for any positive γ.

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1 Introduction

In this paper we study the asymptotic behavior of a system of interacting point processes introduced in [8], which is aimed to offer a good model for neural interactions.

Roughly speaking, the model consists of a graph which nodes represents neurons and the edges the synapses between them. Each neuron has a *membrane potential* which evolves in time. The *spiking activity* of each neuron is represented by a continuous time point process, which rate evolves in time with respect to the membrane potential of the neuron. Moreover all neurons are also exposed to a *leakage effect*, which consists in a sudden loss of membrane potential occurring at random times driven by a Poisson process of some parameter γ. The membrane potential of each neuron integrates all the spiking activity of its pre-synaptic neurons up to the last event by

which the said neuron has been affected (a leakage or a spike), so that the membrane potential of this neuron is also reset to 0 whenever it spikes.

The idea of using stochastic processes to model biological neural networks dynamics has a now a long history and the reader may refer to [16] for a general historical review. Inside these stochastic models, the subcategory making use of point processes has become increasingly popular during the last decades (considering for example Hawkes processes [19, 24, 25, 23] as well as Wold processes [22]). The model considered in this article has the specificity that the membrane potential is reset to 0 after each spike, which made it closer to the model introduced in [20] for example. Moreover it implies that each of the spiking point processes alone has a memory of variable length, which an interesting feature from a biological point of view. It can be seen as a continuous time version of the model introduced in [9], sometime named the Galves-Löcherbach model, or simply GL model. Other versions of this model has been studied since it has been introduced, see for example [21, 11, 18, 4, 5, 6]. We refer to [10] for a general review.

Two crucial area of research in the study of biological neural networks are the question of criticality and metastability [26, 27, 14, 15]. An other interrogation is the topology of the networks. While many complex real life networks have been extensively studied [17], leading to famous models like Small World graphs, a consensus on the topology of neuronal networks has yet to be done. The model we consider here was originally introduced and studied in [8] on the infinite one-dimensional lattice. It was proven there that for this graph, the model is subject to a phase-transition with respect to the leakage parameter γ, namely there exists a critical value γ_c such that for any $\gamma < \gamma_c$ each neuron continues to spike infinitely many times with positive probability while for $\gamma > \gamma_c$ the spiking activity stop with probability 1. The same model was then studied in [3] on the finite one-dimensional lattice of arbitrary length, and it was proven that for γ small enough the renormalized time of extinction of the system is asymptotically memory-less when the number of neurons grows, or in other words, asymptotically exponentially distributed. This feature is a crucial property of metastable dynamics as proposed in the seminal paper [9], which was originally concerned with interacting particle physics.

Here we first prove that for the model defined on the finite one-dimensional lattice of arbitrary length this last result doesn't hold anymore if you take γ big (actually $\gamma > 1$ is sufficient), and that in fact for such a γ the renormalized extinction time is asymptotically deterministic, so that it is highly non-metastable. Then we turn ourself to the finite complete graph, and we prove that, on the contrary, the result of metastability holds for any positive γ in this case. This introduces an interesting dichotomy with respect to metastability between loosely connected graphs, like the one-dimensional lattice, and highly connected ones, such as the complete graph.

This paper is organized as follows. In Section 2 we give a formal definition of the model, and we state the previously proven results as well as the two results that this article is concerned with. In Section 3 we give the proof of the result related to the one-dimensional lattice. In Section 4 we give the proof of the result related to the complete graph.

2   Definition and results

We consider the following model. V is a finite or countable set representing the neurons, and to each $i \in V$ is associated a set $\mathbb{V}_i$ of *presynaptic neurons*. If you consider the elements of V as vertices, and draw and edge from i to j whenever $i \in \mathbb{V}_j$, that is if you define $E = \{(i,j) : i \in \mathbb{V}_j\}$, then you obtain the graph of the network $G = (V, E)$. Each neuron i has a *membrane potential* evolving over time, represented by a stochastic process which takes its values in the set $\mathbb{N}$ of non-negative integers and which is denoted $(X_i(t))_{t \geq 0}$. Moreover each neuron is also associated with a Poisson process $(N_i^\dagger(t))_{t \geq 0}$ of some parameter γ, representing the *leak times*. At any of these leak times the membrane potential of the neuron concerned is reset to 0. Finally an other point process $(N_i(t))_{t \geq 0}$ representing the *spiking times* is also associated to each neuron, which rate at time t is given by $\phi(X_i(t))$, ϕ being some function called the rate function. When a neuron spikes

its membrane potential is reset to 0 and the membrane potential of all of its post-synaptic neurons is increased by one (that is the neurons of the set $\{j : i \in \mathbb{V}_j\}$). All the point processes involved are assumed to be mutually independent.

More formally, beside asking that $(N_i^\dagger(t))_{t \geq 0}$ be a Poisson process of some parameter γ, this is the same as saying that $(N_i(t))_{t \geq 0}$ is the point process characterized by the following equation

$$\mathbb{E}(N_i(t) - N_i(s)|\mathscr{F}_s) = \int_s^t \mathbb{E}(\phi(X_i(u))|\mathscr{F}_s)du \qquad (2.1)$$

where

$$X_i(t) = \sum_{j \in \mathbb{V}_i} \int_{]L_i(t),t[} dN_j(s), \qquad (2.2)$$

$L_i(t)$ being the time of the last event affecting neuron i before time t, that is,

$$L_i(t) := \sup\left\{s \leq t : N_i(\{s\}) = 1 \text{ or } N_i^\dagger(\{s\}) = 1\right\}. \qquad (2.3)$$

$(\mathscr{F}_t)_{t \geq 0}$ is the standard filtration for the point processes involved here. See [8] for more details.

In the present paper the function ϕ will always be the hard-threshold function defined for any $x \in \mathbb{R}$ by $\phi(x) = \mathbb{1}_{x>0}$. For any fixed $t \geq 0$ and $i \in V$ the quantity of interest is then given by $\mathbb{1}_{X_i(t)>0}$, as it corresponds to the infinitesimal spiking rate of neuron i at time t. Depending on whether $\mathbb{1}_{X_i(t)>0}$ is equal to 1 or 0 we will say that neuron i is *active* or *quiescent* respectively. The dynamic described in the first paragraph of this section can then be reformulated as follows. Whenever a spike occurs, the neuron that spiked becomes quiescent, as can be seen from equations (2.3) and (2.2), while its post-synaptic neurons become active if they weren't already, as can be seen from equation (2.2). The choice of the hard-threshold for the function ϕ has however no consequence on the leak times, which effect of getting any active neuron to the quiescent state result from equations (2.3) and (2.2). If for any $i \in V$ and $t \geq 0$ we write $\eta_i(t) = \mathbb{1}_{X_i(t)>0}$, and $\eta(t) = (\eta_j(t))_{j \in V}$, then the resulting process $(\eta(t))_{t \geq 0}$ is an interacting particle system, that is, a Markovian process taking value in $\{0,1\}^V$ (see [12]). His infinitesimal generator (see [8]) is given by

$$\mathscr{L}f(\eta) = \gamma \sum_{i \in V}\left(f(\pi_i^\dagger(\eta)) - f(\eta)\right) + \sum_{i \in V} \eta_i\left(f(\pi_i(\eta)) - f(\eta)\right), \qquad (2.4)$$

where $f : \{0,1\}^V \mapsto \mathbb{R}$ is a cylinder function, γ is a non-negative real number, and the $\pi_i^\dagger$'s and π_i's are maps from $\{0,1\}^V$ to $\{0,1\}^V$ defined for any $i \in V$ as follows:

$$\left(\pi_i^\dagger(\eta)\right)_j = \begin{cases} 0 & \text{if } j = i, \\ \eta_j & \text{otherwise,} \end{cases}$$

and

$$\left(\pi_i(\eta)\right)_j = \begin{cases} 0 & \text{if } j = i, \\ \max(\eta_i, \eta_j) & \text{if } i \in \mathbb{V}_j, \\ \eta_j & \text{otherwise.} \end{cases}$$

The $\pi_i^\dagger$'s correspond to the leakage effect mentioned above, and the π_i's correspond to the spikes. Of course the graph of interaction needs to satisfy some conditions of regularity for this infinitesimal generator to even make sense, but the instantiations that we consider in this article are simple enough to trivially satisfy these conditions. A simple way of reading the generator above is the following. Whenever a neuron is active, it is associated with two independent exponential random clocks of parameter 1 and γ, and it spikes or leaks respectively depending on which one rings first.

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This model was originally introduced in [8], where a specific instantiation was studied, in which the graph G is the infinite one-dimensional lattice, that is the graph characterized by the following.

Assumption 2.1 *Let $V = \mathbb{Z}$ and for all $i \in V$, $\mathbb{V}_i = \{i - 1, i + 1\}$.*

The rate function is the hard-threshold as stated above, and as it will be the case in the rest of the paper we will not indicate it anymore in the sequel. It was proven in [8] that this instantiation is subject to a phase transition. More precisely the following theorem was proven.

Theorem 2.2 *Under Assumption 2.1 and assuming that $X_i(0) \geq 1$ for all $i \in V$, there exists a critical value γ_c for the parameter γ, with $0 < \gamma_c < \infty$, such that for any $i \in \mathbb{Z}$*

$$\mathbb{P}\Big(N_i([0, \infty[) \ < \infty\Big) = 1 \text{ if } \gamma > \gamma_c$$

and

$$\mathbb{P}\Big(N_i([0, \infty[) \ = \infty\Big) > 0 \text{ if } \gamma < \gamma_c.$$

In words there exists a critical value γ_c for the leakage rate which is such that the process dies almost surely above it, and survive with positive probability below it.

For this one-dimensional lattice instantiation let $\xi_i(t) := \mathbb{1}_{X_i(t) > 0}$ for any $i \in \mathbb{Z}$ and $t \geq 0$, so that $\big(\xi(t)\big)_{t \geq 0}$ is the interacting particle system taking value in $\{0, 1\}^{\mathbb{Z}}$ which dynamic is specified by (2.4). Now consider a finite version of the process $\big(\xi(t)\big)_{t \geq 0}$, where the neurons aren't anymore in the infinite lattice but only in a finite portion of it. Fix some integer $N \geq 0$ and let $V_N = [\![-N, N]\!]$ (where $[\![-N, N]\!]$ is a short-hand for $\mathbb{Z} \cap [-N, N]$). For any $i \in V_N$ let the set $\mathbb{V}_{N,i}$ be defined by

$$\mathbb{V}_{N,i} = \begin{cases} \{i - 1, i + 1\} & \text{if } i \in [\![-(N-1), N-1]\!], \\ \{N - 1\} & \text{if } i = N, \\ \{-(N - 1)\} & \text{if } i = -N. \end{cases}$$

We write $\big(\xi_N(t)\big)_{t \geq 0}$ for the finite version of the lattice process, that is to say the process taking value in $\{0, 1\}^{2N+1}$ which dynamic is given by the generator in (2.4) when you replace V by V_N and $\mathbb{V}_i$ by $\mathbb{V}_{N,i}$ for any $i \in V_N$. By convention when no indication is given the initial state of the process is the state where all neurons are actives. Let τ_N denote the time of extinction of this finite process, that is to say the first time at which all neurons are quiescent. Formally

$$\tau_N := \inf\{t \geq 0 : \xi_N(t)_i = 0 \text{ for any } i \in [\![-N, N]\!]\}.$$

It was proven in [3] that the following hold:

Theorem 2.3 *There exists a γ_c' satisfying $\gamma_c' \leq \gamma_c$ such that if $0 < \gamma < \gamma_c'$ then*

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N \to \infty]{\mathscr{D}} \mathscr{E}(1),$$

where $\mathbb{E}$ denotes the expectation, $\mathscr{D}$ denotes a convergence in distribution and $\mathscr{E}(1)$ an exponential random variable of mean 1.

In this article the first result we're aimed to prove is that the result above (Theorem 2.3) doesn't hold anymore if γ is big enough. In fact we prove the following theorem.

Theorem 2.4 *Suppose that $\gamma > 1$. Then the following convergence holds*

$$\frac{\tau_N}{\mathbb{E}(\tau_N)} \xrightarrow[N \to \infty]{\mathbb{P}} 1,$$

where $\mathbb{P}$ denotes a convergence in probability.

This result is in some sense symmetrical to Theorem 2.3. Indeed the later tells us that in a portion of the sub-critical region the time of extinction is asymptotically memory-less, which means that it is highly unpredictable: knowing that the process survived up to time t doesn't give you any information about what should happen after time t. What we prove here is that in a portion of the super-critical regime (indeed $\gamma_c < 1$ as proven in in [8]) the time of extinction is asymptotically constant, so that it is highly predictable.

In the second part of this article we study the behaviour of the same stochastic process when the graph of the network is the complete graph of size N.

For some $N \in \mathbb{N}$ we let $V'_N = [\![1, N]\!]$ and for any $i \in V'_N$ we let $V'_{N,i} = V'_N \setminus \{i\}$. Now write $(\zeta_N(t))_{t \geq 0}$ for the spiking process defined on the finite complete graph of size N, that is to say the process in $\{0, 1\}^N$ which dynamic is given by the generator in (2.4) when you replace V by V'_N and $\mathbb{V}_i$ by $\mathbb{V}'_{N,i}$ for any $i \in V'_N$. In this instantiation every neuron is connected to one another so that when a single active neuron spikes, every other neuron becomes active. Let σ_N denote the time of extinction of this finite process,

$$\sigma_N := \inf\{t \geq 0 : \zeta_N(t)_i = 0 \text{ for any } i \in [\![1, N]\!]\}.$$

The second result we're aimed to prove is that for the complete graph instantiation the result of convergence of the renormalized time of extinction toward an exponential random variable that was proven in the lattice case for small γ now holds for any positive γ. This is the object of the following theorem.

Theorem 2.5 *For any $\gamma > 0$ the following convergence holds*

$$\frac{\sigma_N}{\mathbb{E}(\sigma_N)} \xrightarrow[N \to \infty]{\mathscr{D}} \mathscr{E}(1).$$

3   Result on the one-dimensional lattice

In the following we will repeatedly identify the state space $\{0, 1\}^{\mathbb{Z}}$ with $\mathscr{P}(\mathbb{Z})$, the set of all subsets of $\mathbb{Z}$. Indeed any state η of the process, belonging to $\{0, 1\}^{\mathbb{Z}}$, can be seen as well as an element A of $\mathscr{P}(\mathbb{Z})$, writing $A = \{i \in \mathbb{Z} \text{ such that } \eta_i = 1\}$. For all the processes involved in the proof we will adopt the convention of writing the initial state as a superscript. For example we will write $(\xi^i(t))_{t \geq 0}$ and $(\xi^i_N(t))_{t \geq 0}$ to denote the infinite-lattice process and the finite version respectively, starting at time 0 with only neuron i active and all other neurons quiescent. Note that to avoid an overload in the notation we write i instead of $\{i\}$, which is an abuse of notation. As stated in the previous section, in the absence of superscript the initial state is the state where all neurons are active, in particular $(\xi_N(t))_{t \geq 0}$ will denote the finite version of the process starting from the state $[\![-N, N]\!]$.

The present section contains two propositions. The first one tells us that the time of extinction evolves asymptotically like a logarithm in N, and the second one tells us that the same is true for its expectation. Theorem 2.4 is an immediate consequence of these two propositions. Our proof use some of the ideas developed in [13] concerning the Harris contact process.

Proposition 3.1 *Suppose that $\gamma > 1$. Then there exists a constant $0 < C < \infty$ depending on γ such that the following convergence holds*

$$\frac{\tau_N}{\log(2N + 1)} \xrightarrow[N \to \infty]{\mathbb{P}} C.$$

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Proof: We define the following function

$$t \mapsto f(t) = \log \left(\mathbb{P} \left(\xi^0(t) \neq \emptyset \right) \right).$$

We also define the following constant

$$C' = - \sup_{s > 0} \frac{f(s)}{s}.$$

The first step is to show that the function f is superadditive. For any $s, t \geq 0$ we have the following inequality

$$\mathbb{P} \left(\xi^0(t + s) \neq \emptyset \mid \xi^0(t) \neq \emptyset \right) \geq \mathbb{P} \left(\xi^0(s) \neq \emptyset \right).$$

Indeed saying that the process is still alive at time t is the same as saying that it possesses at least one active neuron at time t, which happens to be the number of active neurons at time 0. Then the inequality follows from the fact that having multiple active neurons in the initial configuration implies having a higher probability to be alive at any given time s compared to an initial configuration with a single active neuron. Moreover this last inequality can be rewritten as follows

$$\mathbb{P} \left(\xi^0(t + s) \neq \emptyset \right) \geq \mathbb{P} \left(\xi^0(t) \neq \emptyset \right) \mathbb{P} \left(\xi^0(s) \neq \emptyset \right),$$

and taking the log gives the superaddtivity we are looking for. Now from a well-known result about superadditive functions, sometimes called the Fekete lemma [2], we get the following convergence

$$\frac{f(t)}{t} \xrightarrow[t \to \infty]{} - C'. \tag{3.5}$$

For any $t > 0$ we also have

$$\mathbb{P} \left(\xi^0(t) \neq \emptyset \right) \leq e^{-C't}. \tag{3.6}$$

Notice that while it is clear that $0 \leq C' < \infty$, it is not obvious that $C' > 0$. We show that it is the case using a coupling with the following branching process. At time 0 there is a single individual. Two independent exponential random clocks of parameter 1 and γ respectively are attached to this individual. If the rate γ clock rings before the other one, then the individual dies. In the contrary case the individual is replaced by two other individuals. Every new individual gets his two own independent exponential clocks of parameter 1 and γ and so on. We denote by $(Z_t)_{t \geq 0}$ the process corresponding to the number of individuals of the population along the time. Note that by hypothesis we have $Z_0 = 1$. The expectation at time t can be explicitly computed (see for example chapter 8 in [1]). For any $\gamma \geq 0$ and $t \geq 0$ we have

$$\mathbb{E} \left(Z_t \right) = e^{-(\gamma - 1)t}. \tag{3.7}$$

The coupling is done as follows, at time 0 the only active neuron in $\xi^0(0)$ is coupled with the only individual in $Z(0)$. By this we mean that if this neuron becomes quiescent then the individual dies, and if the neuron spikes, then the individual is replaced by two new individuals. When a spike occurs, there is three possibilities: two neurons are activated, one neuron is activated, or no neuron is activated, depending on how much neighbours are quiescent. In any of these cases each activated neuron is coupled with a newborn individuals, and any supernumerary newborn individual is given his own independent exponential clocks. At any time $t \geq 0$ we obviously have $|\xi_t^0| \leq Z_t$. Using (3.7) and Markov inequality it follows that

$$\mathbb{P} \left(\xi^0(t) \neq \emptyset \right) \leq \mathbb{P} \left(Z_t \geq 1 \right) \leq e^{-(\gamma - 1)t}.$$

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Then we take the log and divide by t in the previous inequality and we obtain at the limit that $C' \geq \gamma - 1$, and from the assumption that $\gamma > 1$ we get $C' > 0$.

Let us break the suspense and already reveal that the constant C we are looking for is actually simply the inverse of C'. Therefore in order to prove our result we are going to prove that for any $\epsilon > 0$ we have the two following convergences

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} > \epsilon\right) \xrightarrow[N \to \infty]{} 0, \qquad (3.8)$$

and

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} < -\epsilon\right) \xrightarrow[N \to \infty]{} 0. \qquad (3.9)$$

Let us start with (3.8), which is the easiest part. We remark that our process is *additive* in the sense that $\xi_N(t) = \cup_{i \in [\![-N,N]\!]}\xi_N^i(t)$. See [8] for details, it is also an immediate consequence of the graphical construction proposed in [3]. Using additivity and inequality (3.6) we get

$$\mathbb{P}\left(\xi_N(t) \neq \emptyset\right) \leq (2N+1)\mathbb{P}\left(\xi^0(t) \neq \emptyset\right) \leq (2N+1)e^{-C't}. \qquad (3.10)$$

Now, for any $\epsilon > 0$, if you let $t = \left(\frac{1}{C'} + \epsilon\right)\log(2N+1)$ then the following holds

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} > \epsilon\right) = P\left(\xi_N(t) \neq \emptyset\right) \leq e^{-C'\epsilon\log(2N+1)}.$$

Then the fact that $C' > 0$ ensures us that the term on the right-hand side of the inequality goes to 0 as N diverges, which proves (3.8).

It remains to prove (3.9). If for some $N \in \mathbb{N}^*$ we take $t = \left(\frac{1}{C'} - \epsilon\right)\log(2N+1)$, then we can write

$$\mathbb{P}\left(\frac{\tau_N}{\log(2N+1)} - \frac{1}{C'} < -\epsilon\right) = \mathbb{P}\left(\xi_N(t) = \emptyset\right),$$

so that it suffices to show that the right-hand side converges to 0 for this choice of t as N goes to ∞. For reasons that will become clear in a moment we will actually write $t = \frac{1}{C'}\left(1 - \epsilon'\right)\log(2N+1)$, with $\epsilon' = C'\epsilon$.

From (3.5) (and from the fact that $C' > 0$) we get that for any $\epsilon > 0$ and for big enough t

$$\frac{f(t)}{t} \geq -(1 + \epsilon')C',$$

which can be written

$$\mathbb{P}\left(\xi_t^0 = \emptyset\right) \leq 1 - e^{-(1+\epsilon')C't}.$$

Therefore, with $t = \frac{1}{C'}\left(1 - \epsilon'\right)\log(2N+1)$ and N big enough we have

$$\mathbb{P}\left(\xi_t^0 = \emptyset\right) \leq 1 - \frac{1}{(2N+1)^{1-\epsilon'^2}}. \qquad (3.11)$$

Now for any $k \in \mathbb{Z}$ we define

$$F_k := [\![(2k-1)K\log(2N+1), (2k+1)K\log(2N+1)]\!],$$

where K is some constant depending on N which value will be chosen later in order for $K\log(2N+1)$ to be an integer. We then consider a modification of the process $(\xi_N(t))_{t \geq 0}$ where all neurons at the border of one of the sub-windows F_k defined above (i.e. all neurons indexed by

$(2k+1)K\log(2N+1)$ for some $k \in \mathbb{Z}$) are fixed in quiescent state and therefore are never allowed to spike. This modified process is denoted $(\widetilde{\xi}_N(t))_{t \geq 0}$. We also define the following configuration

$$A_N := \left\{ 2kK\log(2N+1) \text{ for } k \in \mathbb{Z} \cap \left[-\frac{N}{2K\log(2N+1)}, \frac{N}{2K\log(2N+1)} \right] \right\}.$$

Notice that the fact that the neurons at the borders of the windows F_k are never allowed to spike makes the evolution of $(\widetilde{\xi}_N(t))_{t \geq 0}$ independent from one window to another. Moreover notice that the integers belonging to A_N are all at the center of one of these windows.

Now for any $t \geq 0$ we define $r_t := \max \xi_t^0$. Considering the spiking process $(\xi_t)_{t \geq 0}$ with no leaking it is easy to see that the right edge r_t can be coupled with an homogeneous Poisson process of parameter 1, that we denote $(M(t))_{t \geq 0}$, in such a way that for any $m \geq 0$

$$\mathbb{P}\left(\sup_{s \leq t} r_s \geq m \right) \leq \mathbb{P}\Big(M(t) \geq m \Big).$$

We have

$$\mathbb{E}\left(e^{M(t)} \right) = e^{t(e-1)},$$

so taking the exponential, using Markov inequality and taking $m = K't$ (where K' is some constant that we are going to fix in a moment) we get

$$\mathbb{P}\left(\sup_{s \leq t} r_s \geq K't \right) \leq e^{t(e-1-K')}$$

$$\leq e^{t(2-K')},$$

where in the last inequality we simply used the fact that $e - 1 < 2$.

Now taking again $t = \frac{1}{C'}(1 - \epsilon')\log(2N+1)$ and $K' = 2(1 + C')$ we get

$$\mathbb{P}\left(\sup_{s \leq t} r_s \geq m \right) \leq e^{-2(1-\epsilon')\log(2N+1)},$$

and assuming without loss of generality that $\epsilon' < \frac{1}{2}$ we get

$$\mathbb{P}\left(\sup_{s \leq t} r_s \geq m \right) \leq \frac{1}{2N+1}. \tag{3.12}$$

It is now possible to fix the value of the constant K we introduced earlier. We take

$$K = \inf\left\{ x \in \mathbb{R} \text{ such that } x \geq \frac{K'}{C'} \text{ and } x\log(2N+1) \in \mathbb{N} \right\}.$$

In words we take K equal to $\frac{K'}{C'}$ and then enlarge it slightly in order for $K\log(2N+1)$ to be an integer. We also define the following event

$$E_t := \left\{ \xi_s^0 \text{ doesn't escape from } [\![-K\log(2N+1), \ldots, K\log(2N+1)]\!] \text{ for any } s \leq t \right\}.$$

Now taking N large enough and $t = \frac{1}{C'}(1 - \epsilon')\log(2N+1)$ we have

$$\mathbb{P}\left(\xi_N(t) = \emptyset\right) \leq \mathbb{P}\left(\widetilde{\xi}_N^{A_N}(t) = \emptyset\right)$$

$$= \mathbb{P}\left(\widetilde{\xi}_N^0(t) = \emptyset\right)^{(2N+1)/(2K\log(2N+1))}$$

$$\leq \left(\mathbb{P}\left(\widetilde{\xi}_N^0(t) = \emptyset \cap E_t\right) + \mathbb{P}\left(E_t^c\right)\right)^{(2N+1)/(2K\log(2N+1))}$$

$$\leq \left(\mathbb{P}\left(\xi^0(t) = \emptyset\right) + \mathbb{P}\left(E_t^c\right)\right)^{(2N+1)/(2K\log(2N+1))}$$

$$\leq \left(1 - \left(\frac{1}{(2N+1)^{1-\epsilon'2}} - \frac{2}{2N+1}\right)\right)^{(2N+1)/(2K\log(2N+1))}.$$

To obtain the inequality above we used (3.11) and the fact that inside E_t the process $(\widetilde{\xi}_N^0(s))_{0 \leq s \leq t}$ evolves just like $(\xi_s^0)_{0 \leq s \leq t}$, which allows us to bound $\mathbb{P}\left(\xi_N^0(t) = \emptyset\right)$, and we used (3.12) to bound $\mathbb{P}\left(E_t^c\right)$.

Finally we let

$$a_N = \frac{1}{(2N+1)^{1-\epsilon'2}} - \frac{2}{2N+1},$$

and

$$b_N = \frac{2N+1}{2K\log(2N+1)},$$

so that the last bound can be written $(1 - a_N)^{b_N}$. Then

$$(1 - a_N)^{b_N} = e^{b_N \log(1-a_N)} \leq e^{-b_N a_N},$$

and since $a_N b_N \xrightarrow[N \to \infty]{} \infty$, it follows that $e^{-b_N a_N}$ goes to 0 as N goes to ∞.

$\square$

The last step consists in showing that the same convergence holds for the expectation, which is the object of the following proposition.

Proposition 3.2 *Suppose that $\gamma > 1$. Then the following convergence holds*

$$\frac{\mathbb{E}\left(\tau_N\right)}{\log(2N+1)} \xrightarrow[N \to \infty]{} C,$$

where C is the same constant as in Proposition 3.1.

Proof: It is well-known that the fact that a sequence of random variables $(X_n)_{n \in \mathbb{N}}$ converges in probability to some random variable X doesn't necessarily implies that $\mathbb{E}(X_n) \xrightarrow[n \to \infty]{} \mathbb{E}(X)$. Nonetheless this implication holds true with the additional assumption that the sequence is uniformly integrable (see for example Theorem 5.5.2 in [7] page 259), i.e. if the following holds

$$\lim_{M \to \infty} \left(\sup_{n \in \mathbb{N}} \mathbb{E}\left(|X_n| \mathbb{1}_{\{|X_n| > M\}}\right)\right) = 0.$$

It is therefore sufficient to show that $\left(\tau_N / \log(2N+1)\right)_{N \in \mathbb{N}^*}$ is uniformly integrable, and the result will follows from Proposition 3.1. For some $M > 0$ and some $N \in \mathbb{N}^*$ it is easy to see that we have the following

$$\mathbb{E}\left(\frac{\tau_N}{\log(2N+1)}\mathbb{1}_{\{\frac{\tau_N}{\log(2N+1)}>M\}}\right) = \int_0^\infty \mathbb{P}\left(\frac{\tau_N}{\log(2N+1)}>\max(t,M)\right)dt.$$

Now using inequality (3.10) and the previously proven fact that $C' > 0$ when $\gamma > 1$ we have the following

$$\int_0^\infty \mathbb{P}\left(\frac{\tau_N}{\log(2N+1)}>\max(t,M)\right)dt$$
$$= \int_0^M \mathbb{P}\left(\frac{\tau_N}{\log(2N+1)}>M\right)dt + \int_M^\infty \mathbb{P}\left(\frac{\tau_N}{\log(2N+1)}>t\right)dt$$
$$\leq (2N+1)\left[\int_0^M e^{-C'\log(2N+1)M}dt + \int_M^\infty e^{-C'\log(2N+1)t}dt\right]$$
$$= (2N+1)^{1-C'M}\left[M+\frac{1}{C'\log(2N+1)}\right],$$

where C' is the same constant as in the previous proof. Without loss of generality we assume that $M > \frac{1}{C'}$, so that the bound above is decreasing in N, from what we get

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\left(\frac{\tau_N}{\log(2N+1)}\mathbb{1}_{\{\frac{\tau_N}{\log(2N+1)}>M\}}\right) \leq 3^{1-C'M}\left[M+\frac{1}{C'\log(3)}\right]. \qquad (3.13)$$

Finally the right-hand side of inequality (3.13) goes to 0 when M goes to ∞, which ends the proof. $\qquad\square$

4 The complete graph case

Recall that $(\zeta_N(t))_{t\geq 0}$ denotes the analog of $(\xi_N(t))_{t\geq 0}$ in the complete graph setting, and that σ_N denotes its time of extinction. We adopt the same conventions as in the previous section regarding the notation, identifying $\{0,1\}^{[\![1,N]\!]}$ with $\mathscr{P}([\![1,N]\!])$ and indicating the initial state as a superscript when it is different from the whole space $[\![1,N]\!]$. This convention applies to the time of extinction as well. Notice that in the complete graph setting it doesn't really make sense to consider an infinite process anymore, as any neuron would be activated by a spike infinitely many times in any time interval for example, moreover the minimal conditions for the existence of an infinitesimal generator are not satisfied. Luckily we wont need any infinite process in the course of the proof.

We fix some $\gamma > 0$. We notice that the time of extinction σ_N is almost surely finite since the state where all neurons are quiescent is a trap and since it can always be reached from any configuration if all neurons leaks one after the other, an event which probability can be bounded below by $(\gamma/(1+\gamma))^N$. Moreover it is clear that $\mathbb{P}(\sigma_N > t)$ is a continuous and decreasing function of t, which converges to 0 when t goes to infinity for the reason we just raised. Therefore we can define β_N to be the unique value in $\mathbb{R}^+$ such that

$$\mathbb{P}(\sigma_N > \beta_N) = e^{-1}.$$

We are going to prove that

$$\frac{\sigma_N}{\beta_N} \xrightarrow[N\to\infty]{\mathscr{D}} \mathscr{E}(1). \qquad (4.14)$$

The result will then follow as can be seen at the end of the proof. The reason for the introduction of β_N is that it will allow us to prove that the mean of the exponential random variable is indeed 1.

We first prove that for any $s, t > 0$ the following holds

$$\lim_{N \to \infty} \left| \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s + t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s \right) \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right| = 0. \tag{4.15}$$

In our complete graph setting there is no spatial dependence between the neurons like in the lattice setting, so that the law of the time of extinction is impacted by the initial state only through its cardinal. In particular, for any $1 \leq k \leq N$ and any $A \in \mathscr{P}(\llbracket 1, N \rrbracket)$ of size k, σ_N^A has the same law as $\sigma_N^{\llbracket 1, k \rrbracket}$. Using the Markov property and this last remark we have

$$\left| \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s + t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s \right) \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right|$$

$$= \sum_{k=1}^{N} \left| \mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right| \cdot \mathbb{P}\left(|\zeta_N(\beta_N s)| = k \right)$$

$$\leq \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P}\left(|\zeta_N(\beta_N s)| = k \right) + \sum_{k=\lceil \frac{N}{2} \rceil}^{N} \left| \mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right|$$

We fix $\epsilon > 0$. In order to prove the desired result we show that we can find N big enough such that

$$\sum_{k=\lceil \frac{N}{2} \rceil}^{N} \left| \mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right| < \epsilon, \tag{4.16}$$

and

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P}\left(|\zeta_N(\beta_N s)| = k \right) < \epsilon. \tag{4.17}$$

We start with (4.16), which is the easiest part. For any $\lceil \frac{N}{2} \rceil \leq k \leq N$ we denote by E_k the event in which every active neuron in the process starting from $\llbracket 1, k \rrbracket$ becomes quiescent before any of them is affected by a spike. Notice that on the complementary event E_k^c there is a spike affecting the process at some point, and that the process starting from $\llbracket 1, k \rrbracket$ and the process starting from $\llbracket 1, N \rrbracket$ become equal at this point (both will be in the state $\llbracket 1, N \rrbracket \setminus \{i\}$, where i is the neuron that just spiked). It follows that we have

$$\mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \,\Big|\, E_k^c \right) = \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \,\Big|\, E_k^c \right).$$

From this we get that, for any $\lceil \frac{N}{2} \rceil \leq k \leq N$,

$$\left| \mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \right) \right|$$

$$= \left| \mathbb{P}\left(\frac{\sigma_N^{\llbracket 1, k \rrbracket}}{\beta_N} > t \,\Big|\, E_k \right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t \,\Big|\, E_k \right) \right| \cdot \mathbb{P}(E_k)$$

$$\leq \mathbb{P}(E_k).$$

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Now since $\mathbb{P}(E_k) = \left(\frac{\gamma}{1+\gamma}\right)^k \leq \left(\frac{\gamma}{1+\gamma}\right)^{\frac{N}{2}}$ for any $k \geq \lceil \frac{N}{2} \rceil$ we get the following bound

$$\sum_{k=\lceil \frac{N}{2} \rceil}^{N} \left| \mathbb{P}\left(\frac{\sigma_N^{[\![1,k]\!]}}{\beta_N} > t\right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right) \right| \leq \frac{N}{2} \left(\frac{\gamma}{1+\gamma}\right)^{\frac{N}{2}},$$

which goes to zero as N goes to infinity, so that we can find some N_1 such that (4.16) is satisfied for any $N \geq N_1$.

We now turn to (4.17). We will use a coupling. Let $(\widetilde{\zeta}_N(t))_{t\geq 0}$ be the process defined as follows. For $t < \sigma_N$ the process is simply equal to $\zeta_N(t)$. Now suppose $i \in [\![1, N]\!]$ is the last neuron active in $(\zeta_N(t))_{t\geq 0}$ before the extinction at σ_N, then when i leaks in $(\zeta_N(t))_{t\geq 0}$, instead of leaking in $(\widetilde{\zeta}_N(t))_{t\geq 0}$ it spikes. Then the dynamic of $(\widetilde{\zeta}_N(t))_{t\geq 0}$ is the same as the dynamic of the process $(\zeta_N(t))_{t\geq 0}$ with the only difference that whenever there is only one neuron remaining it doesn't leak, and spike at rate $1 + \gamma$, so that there is no extinction for this stochastic process. The reason for asking that the last neuron spike at rate $1 + \gamma$ instead of simply 1 is that if X and Y are two independent random variable exponentially distributed with rate 1 and γ respectively, then one can easily compute that the law of X conditioned on $X \leq Y$ is an exponential distribution of rate $1 + \gamma$. Now for any $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, we have

$$\begin{aligned}
\mathbb{P}\Big(|\zeta_N(\beta_N s)| = k\Big) &= \mathbb{P}\Big(|\zeta_N(\beta_N s)| = k \mid \sigma_N > \beta_N s\Big)\mathbb{P}\Big(\sigma_N > \beta_N s\Big) \\
&= \mathbb{P}\Big(|\widetilde{\zeta}_N(\beta_N s)| = k \mid \sigma_N > \beta_N s\Big)\mathbb{P}\Big(\sigma_N > \beta_N s\Big) \qquad (4.18) \\
&\leq \mathbb{P}\Big(|\widetilde{\zeta}_N(\beta_N s)| = k\Big),
\end{aligned}$$

so that it will be sufficient to prove that for N big enough we have

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P}\left(|\widetilde{\zeta}_N(\beta_N s)| = k\right) < \epsilon. \qquad (4.19)$$

One can easily see that the process $(|\widetilde{\zeta}_N(t)|)_{t\geq 0}$, that is to say the process counting the number of particles at any time t, is a Markov jump process taking value in $[\![1, N]\!]$, which transition diagram is given in Figure 1.

We would like to compute an invariant measure for this chain, so that we need to solve the following equation for μ.

$$\mu Q = 0,$$

where Q is the transition intensities matrix, given by

$$Q = \begin{pmatrix}
-(1+\gamma) & 0 & 0 & \cdots & 0 & 0 & 1+\gamma & 0 \\
2\gamma & -2(1+\gamma) & 0 & \cdots & 0 & 0 & 2 & 0 \\
0 & 3\gamma & -3(1+\gamma) & \cdots & 0 & 0 & 3 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (N-2)\gamma & -(N-2)(1+\gamma) & N-2 & 0 \\
0 & 0 & 0 & \cdots & 0 & (N-1)\gamma & -(N-1)\gamma & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & N(1+\gamma) & -N(1+\gamma)
\end{pmatrix}.$$

This is the same as the following system of linear equations.

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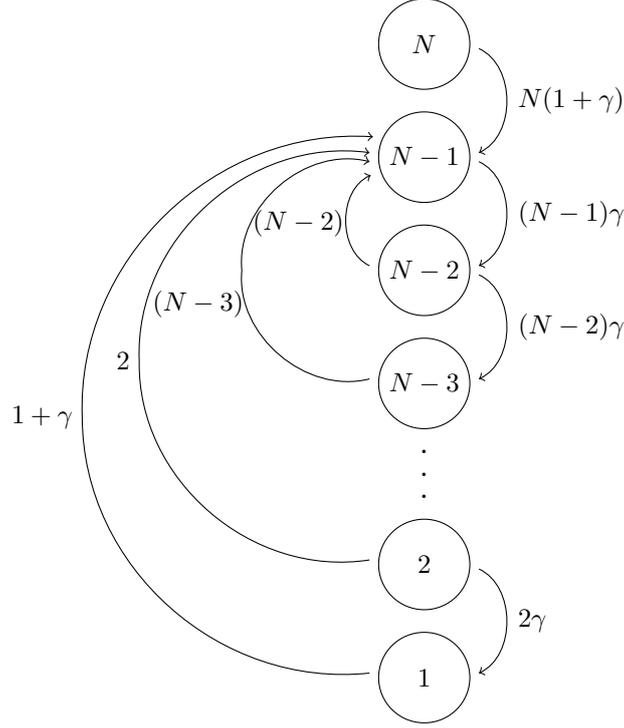
Figure 1: The transition diagram of the Markov jump process $(\widetilde{\zeta}_N(t))_{t \geq 0}$.

$$
\begin{cases}
-(1 + \gamma)\mu_1 + 2\gamma\mu_2 = 0, \\
-2(1 + \gamma)\mu_2 + 3\gamma\mu_3 = 0, \\
\qquad \vdots \\
-(N - 2)(1 + \gamma)\mu_{N-2} + (N - 1)\gamma\mu_{N-1} = 0, \\
(1 + \gamma)\mu_1 + 2\mu_2 + 3\mu_3 + \ldots + (N - 2)\mu_{N-2} - (N - 1)\gamma\mu_{N-1} + N(1 + \gamma)\mu_N = 0, \\
-N(1 + \gamma)\mu_N = 0.
\end{cases}
\tag{4.20}
$$

Solving the system from top to bottom, we get that, for any $n \in [\![2, N - 1]\!]$

$$
\mu_n = \frac{(1 + \gamma)^{n-1}}{n\gamma^{n-1}}\mu_1.
\tag{4.21}
$$

Moreover $\mu_N = 0$. Now from (4.21) and from the fact that the elements of μ need to sum up to 1 we get

$$
\mu_1 = \left(\sum_{n=1}^{N-1} \frac{(1 + \gamma)^{n-1}}{n\gamma^{n-1}} \right)^{-1}.
$$

Disregarding all terms in the sum but the last one we obtain the following bound

$$
\mu_1 < N \left(\frac{1 + \gamma}{\gamma} \right)^{2-N}.
$$

Hence, for any $1 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor$

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$$\mu_k < N \left(\frac{1+\gamma}{\gamma} \right)^{k+1-N}.$$

As a consequence,

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mu_k < \frac{N^2}{2} \left(\frac{1+\gamma}{\gamma} \right)^{1-\frac{N}{2}}. \tag{4.22}$$

Let $\left(\widetilde{\zeta}_N^\mu(t) \right)_{t \geq 0}$ denote the process which initial state is chosen according to the invariant measure μ. By this we mean that a value $k \in [\![1, N]\!]$ is sorted according to the invariant measure, and that the process then start from the initial state $[\![1, k]\!]$. For any $t \geq 0$ we have the following inequality:

$$\mu_{N-1} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P} \left(\left| \widetilde{\zeta}_N^{[\![1,N-1]\!]}(t) \right| = k \right) \leq \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P} \left(|\widetilde{\zeta}_N^\mu(t)| = k \right). \tag{4.23}$$

This last inequality gets us closer to our goal but it is still not exactly what we need as the left hand side involves the process starting from $[\![1, N-1]\!]$ while we would like it to start from the full initial configuration $[\![1, N]\!]$. This little problem is solved as follows. Let T_N be the time of the first jump of the process $(\widetilde{\zeta}_N(t))_{t \geq 0}$, that is to say

$$T_N := \inf\{t \geq 0 : |\widetilde{\zeta}_N(t)| \neq N\}.$$

Then, for any $t \geq 0$ and $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$, the following holds

$$\mathbb{P} \left(\left| \widetilde{\zeta}_N(t) \right| = k \right) = \mathbb{P} \left(\left| \widetilde{\zeta}_N^{[\![1,N-1]\!]} \left((t - T_N)^+ \right) \right| = k \right), \tag{4.24}$$

where $(t - T_N)^+$ stands for $\max(0, t - T_N)$. This last inequality is obtained from Markov property and the fact that whenever $T_N > t$ the events we look at are both of probability 0 for the k we consider (assuming that $N \geq 3$).

Now from (4.22), (4.23) and (4.24), we obtain

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P} \left(\left| \widetilde{\zeta}_N(\beta_N s) \right| = k \right) = \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P} \left(\left| \widetilde{\zeta}_N^{[\![1,N-1]\!]} \left((\beta_N s - T_N)^+ \right) \right| = k \right)$$

$$\leq \frac{1}{\mu_{N-1}} \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P} \left(\left| \widetilde{\zeta}_N^\mu \left((\beta_N s - T_N)^+ \right) \right| = k \right)$$

$$\leq \frac{1}{\mu_{N-1}} \frac{N^2}{2} \left(\frac{1+\gamma}{\gamma} \right)^{1-\frac{N}{2}}.$$

Moreover from the penultimate line of (4.20) we get

$$\mu_{N-1} > \frac{\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_{N-2}}{\gamma(N-1)}.$$

Furthermore, assuming that N is sufficiently big for $\gamma(N-1)$ to be greater than one, and using again the fact that the elements of μ need to sum up to 1, we obtain

$$\mu_{N-1} + \frac{\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_{N-2}}{\gamma(N-1)} \geq \frac{1}{\gamma(N-1)}.$$

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Combining the two previous equations we have

$$\mu_{N-1} \geq \frac{1}{2\gamma(N-1)}.$$

Hence, we finally get

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \mathbb{P}\left(|\widetilde{\zeta}_N(\beta_N s)| = k\right) \leq 2\gamma(N-1)\frac{N^2}{2}\left(\frac{1+\gamma}{\gamma}\right)^{1-\frac{N}{2}}$$

$$\leq (1+\gamma)N^3\left(\frac{1+\gamma}{\gamma}\right)^{-\frac{N}{2}}.$$

And the last bound goes to zero as N goes to infinity, so that we can find some N_2 such that (4.17) is satisfied for any $N \geq N_2$. Finally (4.16) and (4.17) are both satisfied for $N \geq \max(N_1, N_2)$ so that the proof of (4.15) is over.

Notice that from (4.15) alone the survival function of the limiting distribution isn't necessarily $t \mapsto e^{-t}$, as it could as well be identically equal to 1 or identically equal to 0. Hopefully the definition of β_N rules out the two last cases, so that we indeed have

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right) = e^{-t}. \tag{4.25}$$

It remains to prove that we can replace β_N by $\mathbb{E}(\sigma_N)$. From the following identity,

$$\left[\mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s+t\right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s\right)\mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right)\right]$$

$$= \sum_{k=1}^{N}\left[\mathbb{P}\left(\frac{\sigma_N^{[\![1,k]\!]}}{\beta_N} > t\right) - \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right)\right] \cdot \mathbb{P}(|\zeta_N(\beta_N s)| = k),$$

and from the fact that for any $t \geq 0$ and $k \in [\![1, N]\!]$ we have

$$\mathbb{P}\left(\frac{\sigma_N^{[\![1,k]\!]}}{\beta_N} > t\right) \leq \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right),$$

it follows that

$$\mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s+t\right) \leq \mathbb{P}\left(\frac{\sigma_N}{\beta_N} > s\right)\mathbb{P}\left(\frac{\sigma_N}{\beta_N} > t\right).$$

Moreover from the definition of β_N we have that for any $n \in \mathbb{N}$

$$\mathbb{P}\left(\frac{\tau_N}{\beta_N} > n\right) \leq e^{-n}.$$

Therefore, for any $t \geq 0$ we have

$$\mathbb{P}\left(\frac{\tau_N}{\beta_N} > t\right) \leq e^{-\lfloor t \rfloor}.$$

Using the Dominated Convergence Theorem and (4.25) we finally get

$$\lim_{N \to \infty} \frac{\mathbb{E}\left(\tau_N\right)}{\beta_N} = \lim_{N \to \infty} \int_0^\infty \mathbb{P}\left(\frac{\tau_N}{\beta_N} > t\right) dt$$
$$= \int_0^\infty \lim_{N \to \infty} \mathbb{P}\left(\frac{\tau_N}{\beta_N} > t\right) dt$$
$$= \int_0^\infty e^{-t} dt$$
$$= 1.$$

And the proof is over.

Acknowledgements

References

[1] R.B. SCHINAZI (2014). "Classical and Spatial Stochastic Processes With Applications to Biology". *Birkhäuser Basel, ed. 2.*

[2] M. FEKETE (1923). "Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten". *Mathematische Zeitschrift, Vol. 17, 228–249.*

[3] M. ANDRE (2019). "A Result of Metastability for an Infinite System of Spiking Neurons". *Journal of Statistical Physics, vol. 177, pp 984–1008.*

[4] A. DE MASI, A. GALVES, E. LÖCHERBACH and E. PRESUTTI (2015). "Hydrodynamic Limit for Interacting Neurons". *Journal of Statistical Physics, Vol. 158, Issue 4, pp 866–902*

[5] A. DUARTE and G. OST (2016). "A Model for Neural Activity in the Absence of External Stimuli". *Markov Processes And Related Fields, Vol. 22, pp 37-52.*

[6] N. FOURNIER and E. LÖCHERBACH (2016). "On a Toy Model of Interacting Neurons". *Annales de l'Institut Henri Poincaré, Probababilité et Statistiques, Vol. 52, Number 4, pp 1844-1876.*

[7] R. DURRETT (2010). "Probability: Theory and Examples". *Cambridge University Press, Fourth Edition.*

[8] P.A. FERRARI, A. GALVES, I. GRIGORESCU and E. LÖCHERBACH (2018). "Phase transition for infinite systems of spiking neurons". *Journal of Statistical Physics, Vol.172, pp 1564–1575.*

[9] A. GALVES and E. LÖCHERBACH (2013). "Infinite Systems of Interacting Chains with Memory of Variable Length". *Journal of Statistical Physics, Vol. 151, pp 896–921*

[10] A. GALVES and E. LÖCHERBACH (2015). "Modeling Networks of Spiking Neurons as Interacting Processes with Memory of Variable Length". *Journal de la Société Française de Statistique, Vol. 157, No. 1, pp 17-32*

[11] A. GALVES, E. LÖCHERBACH and C. POUZAT (2019) "A System of Interacting Neurons with Short Term Synaptic Facilitation." *Journal of Statistical Physics, vol. 178, pp 869–892*

[12] T. M. LIGGETT (1985). "Interacting Particle Systems". *Grundlehren der mathematischen Wissenschaften, 276.*

[13] R. DURRETT and X.F. LIU (1988). "The Contact Process on a Finite Set". *The Annals of Probability, Vol. 16, No. 3, 1158-1173.*

[14] J. M. BEGGS (2008). "The criticality hypothesis: how local cortical networks might optimize information processing." *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 366(1864), pp 329-343.*

[15] G. WERNER (2007). "Metastability, criticality and phase transitions in brain and its models". *Biosystems Vol. 90, Issue 2*

[16] H. TUCKWELL (1989). "Stochastic processes in the neurosciences". *Vol. 56. Siam.*

[17] S. BOCCALETTI, V. LATORA, Y. MORENO, M. CHAVEZ, D.U. HWANG (2006). "Complex networks: Structure and dynamics." *Physics reports, 424(4-5), pp 175-308.*

[18] I. PAPAGEORGIOU (2020) "Modified Log-Sobolev Inequality for a Compact Pure Jump Markov Process with Degenerate Jumps." *To be published in Journal of Statistical Physics. Pre-print arXiv:1907.09012.*

[19] A. G. HAWKES (1971) "Point Spectra of some Mutually Exciting Point Processes." *Journal of the Royal Statistical Society, vol. 33, n. 3, pp 438-443.*

[20] D. R. BRILLINGER (1988) "Maximum likelihood analysis of spike trains of interacting nerve cells." *Biological cybernetics, vol. 59, n. 3, pp 189-200.*

[21] A. DUARTE and G. OST (2016) "A Model for Neural Activity in the Absence of External Stimuli." *Markov Processes Related Fields, vol. 22, n. 1, pp 35-52.*

[22] R. E. KASS and V. VENTURA (2001) "A spike-train probability model." *Neural computation. vol. 13, n. 8, pp 1713-1720.*

[23] N. R. HANSEN, P. REYNAUD-BOURET and V. RIVOIRARD (2015) "Lasso and probabilistic inequalities for multivariate point processes." *Bernoulli, vol. 21, n. 1, pp 83-143.*

[24] J. CHEVALLIER, M. J. CACERES, M. DOUMIC and P. REYNAUD-BOURET (2015) "Microscopic approach of a time elapsed neural model." *Mathematical Models and Methods in Applied Sciences, vol. 25, n. 14, pp 2669-2719.*

[25] E. S. CHORNOBOY, L. P. SCHRAMM and A. F. KARR (1988) "Maximum likelihood identification of neural point process systems." *Biological Cybernetics, vol. 59, pp 265–275.*

[26] G. DECO, M. L. KRINGELBACH, V.K. JIRSA and P. RITTER (2017) "The dynamics of resting fluctuations in the brain: metastability and its dynamical cortical core." *Scientific Reports, vol. 7.*

[27] J. A. ROBERT, L. L. GOLLO, R. G. ABEYSURIYA, G, ROBERTS, P. B. MITCHELL, M. W. WOOLRICH, and M. BREAKSPEAR (2019) "Metastable Brain Waves" *Nature Communications, vol. 10.*