Minimum Eccentricity Shortest Path Problem: an Approximation Algorithm and Relation with the k-Laminarity Problem

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Abstract. The Minimum Eccentricity Shortest Path (MESP) Problem consists in determining a shortest path (a path whose length is the distance between its extremities) of minimum eccentricity in a graph. It was introduced by Dragan and Leitert [9] who described a linear-time algorithm which is an 8-approximation of the problem. In this paper, we study deeper the double-BFS procedure used in that algorithm and extend it to obtain a linear-time 3-approximation algorithm. We moreover study the link between the MESP problem and the notion of laminarity, introduced by Völkel *et al* [12], corresponding to its restriction to a diameter (*i.e.* a shortest path of maximum length), and show tight bounds between MESP and laminarity parameters.

1 Introduction

For both graph classification purposes and applications, it is an important issue to determine to which extent a graph can be summarized by a path. Different path constructions and metrics to characterize how far the graph is from the constructed path can be used, for example path-decompositions and path-width [11] or path-distance-decompositions and path-distance-width [13]. Another approach, on which we focus in this article, is to characterize the graph by a spine defined by one of its paths.

This problem was first studied in terms of domination, that is finding a path such that every vertex in the graph belongs to or has a neighbor in the path. Several graphs classes were defined in terms of dominating paths. [7] studies the graphs for which the dominating path is a diameter. [8] introduces dominating pairs, that is vertices such that every path linking them is dominating. Graphs such that short dominating paths are present in all induced subgraphs are characterized in [2]. Linear-time algorithms to find dominating paths or dominating vertex pairs were also developed for AT-free graphs [4,6].

Dominating paths do not exist however in every graph and have no associated metric to measure the distance from the graph to the path. A natural extension of the notion of domination is the notion of k-coverage for a given integer k, defined by the fact that a path k-covers the graphs if every vertex is at distance at most k from the path. The smallest k such that a path k-covers the graph is then a metric as desired.

In the present paper, we study the latter problem in which the covering path is required to be a shortest path between its end-vertices. It was introduced in [9] as the Minimum Eccentricity Shortest Path Problem, and shown to be linked to the minimum line distortion problem [14].

The MESP problem is also closely related to the notion of k-laminar graphs introduced in [12], in which the covering path is required to be a diameter.

The MESP problem, as well as determining if a graph is k-laminar for a given k, are NP-hard [9,12]. However, Dragan and Leitert [9] develop a 2-approximation algorithm for MESP of time complexity $\mathcal{O}(n^3)$, a 3-approximation algorithm in $\mathcal{O}(nm)$ and a linear 8-approximation. The latter is extremely simple as it consists in a double-BFS procedure.

Roadmap

In this paper, we introduce a different analysis of the double-BFS procedure and prove that it is in fact a 5-approximation algorithm, and that the bound is tight. We then develop the idea of this algorithm and reach a 3-approximation, which still runs in linear time. Finally, we establish bounds relating the MESP problem and the notion of laminarity.

Definitions and Notations

Through this paper $G = (V, E)$ denotes a finite connected undirected graph. A *shortest path* between two vertices u and v is a path whose length is minimal among all u, v-paths. This length (counting edges) is the *distance* $d(u, v)$. Depending on the context, we consider a path either as a sequence, or as a set of vertices. The *distance* $d(v, S)$ between a vertex v and a set S is smallest distance between v and a vertex from S.

The *eccentricity* $ecc(S)$ of a set S is the largest distance between S and any vertex of G.

The maximal eccentricity of any singleton $\{v\}$, or equivalently the largest distance between two vertices, denoted here $diam(G)$, is often called the diameter of the graph, but for clarity in this paper *a diameter* is always a shortest path of maximum length, *i.e.* a shortest path of length $diam(G)$, and not its length.

2 Double-BFS is a 5-Approximation Algorithm

Let us define the problem we are interested in:

Definition 1 (Minimum Eccentricity Shortest Path Problem (MESP)).
Given a graph G, find a shortest path P such that, for every shortest path Q, $ecc(P) \leq ecc(Q)$.

$k(G)$ denotes the eccentricity of a MESP of G.

Theorem 1 (Dragan and Leitert [9]). *Computing $k(G)$ or finding a MESP are NP-complete problems.*

It is therefore worth using polynomial-time approximation algorithms. We say that an algorithm is an α-approximation of the MESP if every path output by this algorithm is a shortest path of eccentricity at most $\alpha k(G)$.

Double-BFS is a widely used tool for approximating $diam(G)$ [3]. It simply consists in the following procedure:

1. Pick an arbitrary vertex r
2. Perform a BFS (Breadth-First Search) starting at r and ending at x. x is thus one of the furthest vertices from r.
3. Perform a BFS (Breadth-First Search) starting at x and ending at y.

The output of the algorithm is the path from x to y, called a *spread path*, while its extremities (x, y) are called a *spread pair*. A folklore result is that the distance between x and y 2-approximates the diameter of G. As noted by Dragan and Leitert, Double-BFS may also be used for approximating MESP: they have shown in [9] that any spread path is an 8-approximation of the MESP problem.

The first result of the present paper is that any spread path is in fact a 5-approximation of the MESP problem and that the bound is tight. But before we prove this result (Theorem 2), let us give the key lemma used for proving our three theorems:

Lemma 1. *Let G be a graph having a shortest path $v_0, v_1...v_t$ of eccentricity k.*
Let $P = x_0, x_1, ...x_s$ be a shortest path of G.
Let i_{min}^P (resp. i_{max}^P) be the smallest (resp. largest) integer such that $v_{i_{min}^P}$ (resp. $v_{i_{max}^P}$) is at distance at most k of P.

For every integer i such that $i_{min}^P \leq i \leq i_{max}^P$, v_i is then at distance at most $2k$ from P.

Subsequently, every vertex v of G at distance at most k from the subpath between $v_{i_{min}^P}$ and $v_{i_{max}^P}$ is at distance at most $3k$ of P.

One may think, at first glance, that this lemma looks similar to the following:

Lemma 2 (from Dragan *et al.* [9]). *If G has a shortest path of eccentricity at most k from s to t, then every path Q with s in Q and $d(s,t) \leq max_{v \in Q} d(s,v)$ has eccentricity at most $3k$.*

The difference lies in the fact that the k in Lemma 2 is specific to the given couple of vertices (s,t) while the k in Lemma 1 is global. On the other hand, Lemma 2 gives a bound on the eccentricity of a path with respect to the whole graph, while Lemma 1 only guarantees an eccentricity for a defined subgraph.

Proof (of Lemma 1). The second assertion of the lemma is straightforward given the first one. To prove the latter, we define, for all l between 0 and s, the subpath $P_l = x_0, x_1...x_l$.

Let us show by induction on l that for all i between $i_{min}^{P_l}$ and $i_{max}^{P_l}$, v_i is at distance at most $2k$ of P_l.

- $l = 0$, $P_0 = x_0$.
Using the triangle inequality:

$$d(v_{i_{min}^{P_0}}, v_{i_{max}^{P_0}}) \leq d(v_{i_{min}^{P_0}}, x_0) + d(x_0, v_{i_{max}^{P_0}}) \leq 2k \tag{1}$$

Hence, for all i between $i_{min}^{P_0}$ and $i_{max}^{P_0}$,

$$d(v_{i_{min}^{P_0}}, v_i) \leq k \text{ or } d(v_{i_{max}^{P_0}}, v_i) \leq k \tag{2}$$

The result is thus verified for $l = 0$.

- Let l in $(1...s)$ such that the property if verified for $l - 1$.
For all i between $i_{min}^{P_{l-1}}$ and $i_{max}^{P_{l-1}}$, v_i is at distance at most $2k$ of P_{l-1} by the induction hypothesis. Hence, v_i is at distance at most $2k$ of P_l.
Moreover,

$$d(v_{i_{max}^{P_{l-1}}}, v_{i_{max}^{P_l}}) \leq d(v_{i_{max}^{x_{l-1}}}, v_{i_{max}^{x_l}}) \tag{3}$$

and by the triangle inequality:

$$d(v_{i_{max}^{x_{l-1}}}, v_{i_{max}^{x_l}}) \leq d(v_{i_{max}^{x_{l-1}}}, x_{l-1}) + d(x_{l-1}, x_l) + d(x_l, v_{i_{max}^{x_l}}) \leq 2k + 1 \tag{4}$$

As the sub-path of P between $v_{i_{max}^{P_{l-1}}}$ and $v_{i_{max}^{P_l}}$ is a shortest path, it follows that for all i between $i_{max}^{P_{l-1}}$ and $i_{max}^{P_l}$,

$$d(v_{i_{max}^{P_{l-1}}}, v_i) \leq k \text{ or } d(v_{i_{max}^{P_l}}, v_i) \leq k, \tag{5}$$

meaning that v_i is at distance at most $2k$ of P_{l-1} or of x_l.

A similar proof shows that for all i between $i_{min}^{P_l}$ and $i_{min}^{P_{l-1}}$, v_i is at distance at most $2k$ from P_{l-1} or from x_l.

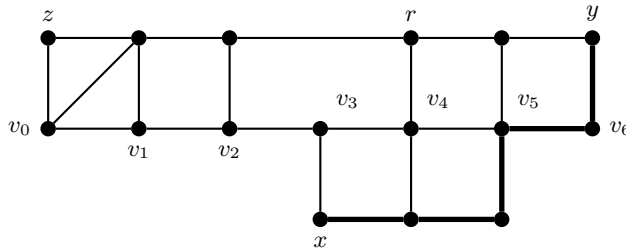The property is verified by induction, and the lemma follows for $l = s$.

Fig. 1. The bound shown in Theorem 2 is tight. Indeed the graph is such that $v_0, v_1...v_6$ is a shortest path of eccentricity 1. The vertex z is at distance 5 from the shortest path (shown by thick edges) between x and y computed by double-BFS starting at r.

Theorem 2. *A double-BFS is a linear-time 5-approximation algorithm for the MESP problem.*

Before we prove it, notice that Figure 1 shows that this bound is tight.

Proof. Let k be $k(G)$, $P = v_0, v_1...v_t$ be a MESP (its eccentricity is thus k), and $Q = x, ..., y$ be the result of a double-BFS starting at some arbitrary vertex r, then reaching x, then reaching y. We shall prove that Q is a $5k$-dominating path of G.

Let i (resp. j) be such that v_i (resp. v_j) is at distance at most k of r (resp. x). The following inequalities are verified:

$$d(r, x) \geq d(r, v_t) \geq d(v_i, v_t) - d(r, v_i) \geq d(v_i, v_t) - k \tag{6}$$

$$d(r, x) \leq d(r, v_i) + d(v_i, v_j) + d(v_j, x) \leq d(v_i, v_j) + 2k \tag{7}$$

Combining those inequalities,

$$d(v_i, v_t) - 3k \leq d(v_i, v_j) \tag{8}$$

Similarly:

$$d(v_i, v_0) - 3k \leq d(v_i, v_j) \tag{9}$$

Therefore v_j is at distance at most $3k$ of v_0 or v_t. Without loss of generality, assume that v_j is at distance at most $3k$ of v_0.

Let l be such that v_l is at distance at most k of y. We distinguish two cases:

(i) $l \leq j$:
   Then y is at distance at most $5k$ of x. As y is a vertex most distant from x, x is a $5k$-dominating vertex of the graph. The lemma is then verified.

(ii) $l > j$:

Applying to (x, y) the inequalities established at the beginning of the proof:

$$d(v_j, v_t) - 3k \le d(v_j, v_l) \tag{10}$$

As $l > j$, it follows that:

$$d(v_l, v_t) \le 3k \tag{11}$$

Figure 2 shows the configuration of the graph in that case. The vertices at distance at most k of a vertex v_s such that $s \le j$ (resp. $s \ge l$) are at distance at most $5k$ of x (resp. y).

According to Lemma 1, every vertex v of G at distance at most k of a vertex v_s such that s is between j and l is at distance at most $3k$ of any shortest path between x and y. The lemma is thus verified.
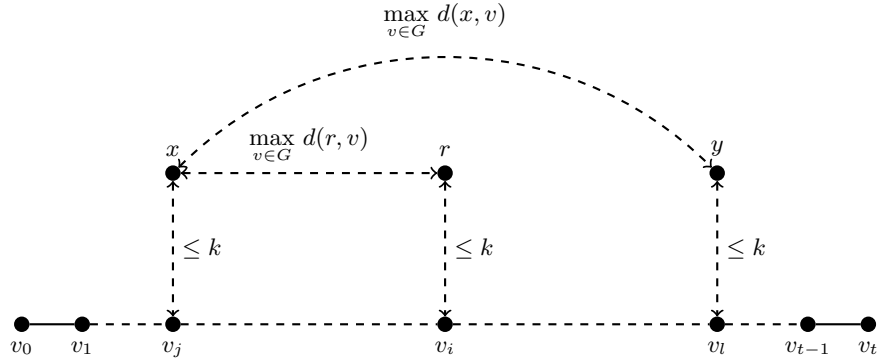

Fig. 2. Notations used in the proof of Theorem 2

3 A 3-Approximation Algorithm

We show now that by using more BFS runs we may obtain a $3k$-approximation of MESP, still in linear time.

Let *bestPath* and *bestEcc* be global variables used as return values for the path and its eccentricity. *bestPath* stores a path and is uninitialized, and *bestEcc* is an integer initialized with $|V(G)|$.

Data: G graph, x,y vertices of G, *step* integer
1 Compute a shortest path Q between x and y;
2 Select a vertex z of G most distant from Q;
3 **if** $d(Q,z) < bestEcc$ **then**
4 | $bestPath \leftarrow Q$;
5 | $bestEcc \leftarrow d(Q,z)$;
6 **end**
7 **if** $step < 8$ **then**
8 | Algorithm3k($G,x,z,step+1$);
9 | Algorithm3k($G,y,z,step+1$);
10 **end**

Algorithm 1: Algorithm3k

Theorem 3. *A 3-approximation of the MESP Problem can be computed in linear time by considering a spread pair (s,l) of G and running Algorithm3k($G,s,l,0$).*

Proof (Correctness). Let G be a graph admitting a shortest path P $= v_0, v_1...v_t$ of eccentricity k.

Let x and y be any vertices of G, $Q_{x,y}$ a shortest path between x and y. Define $i_{min}^{x,y}$ (resp. $i_{max}^{x,y}$) as the smallest (resp. largest) integer such that $v_{i_{min}^{x,y}}$ (resp. $v_{i_{max}^{x,y}}$) is at distance at most k of x or y. Then, by Lemma 1,

$$\text{For all } j \text{ such that } i_{min}^{x,y} - k \leq j \leq i_{max}^{x,y} + k, \quad d(Q_{x,y}, v_j) \leq 2k \qquad (12)$$

Hence, if $i_{min}^{x,y} \leq k$ and $i_{max}^{x,y} \geq t - k$, every vertex of P is at distance at most $2k$ of $Q_{x,y}$ and, as P is of eccentricity k, $Q_{x,y}$ is of eccentricity at most $3k$.

Algorithm3k uses this implication to exhibit a pair x, y such that $Q_{x,y}$ is of eccentricity at most $3k$. Indeed, in each recursive call, one of the following cases holds:

1. the vertex z selected at line 3 is at distance at most $3k$ from $Q_{x,y}$. In that case, *bestPath* will be set to $Q_{x,y}$ unless it already contains a path of even better eccentricity. In any case, the result of the algorithm is a path of eccentricity at most $3k$.
2. the vertex z is at a distance greater than $3k$ of $Q_{x,y}$. Let i_z be such that v_{i_z} is at distance at most k of z. Then, according to Equation (12),

$$i_z \leq i_{min}^{x,y} - k \text{ or } i_z \geq i_{max}^{x,y} + k \qquad (13)$$

   (a) Suppose that $i_z \geq i_{max}^{x,y} + k$. Then, in the case $d(v_{i_{min}^{x,y}}, x) = k$, we get $i_{min}^{x,z} \leq i_{min}^{x,y}$ and $i_{max}^{x,z} \geq i_{max}^{x,y} + k$. And in the case $d(v_{i_{min}^{x,y}}, y) = k$ we get $i_{min}^{x,z} \leq i_{min}^{x,y} - k$ and $i_{max}^{x,z} \geq i_{max}^{x,y}$.

(b) A similar reasoning can be applied if $i_z \leq i_{min}^{x,y} - k$, also yielding to $i_{min}^{x,z} \leq i_{min}^{x,y}$ and $i_{max}^{x,z} \geq i_{max}^{x,y} + k$ or $i_{min}^{x,z} \leq i_{min}^{x,y} - k$ and $i_{max}^{x,z} \geq i_{max}^{x,y}$.

Therefore, either the algorithm already found a path of eccentricity at most $3k$, or it makes one of its two new calls with a couple (x', y') such that the interval $[i_{min}^{x',y'}, i_{max}^{x',y'}]$ contains $[i_{min}^{x,y}, i_{max}^{x,y}]$ but has length increased by at least k.

Consider now a spread pair (s, l) for which Algorithm3k$(G,s,l,0)$ is run. It follows from case (i) and (ii) of the proof of Theorem 2 that

$$i_{min}^{s,l} \leq 5k \text{ and } i_{max}^{s,l} \geq t - 5k \tag{14}$$

At each of the recursive calls, if no path of eccentricity at most $3k$ has already been discovered, one of the new calls expands the interval $[i_{min}^{x,y}, i_{max}^{x,y}]$ length by at least k, while containing the previous interval. As the recursive calls are made until $step = 8$, it follows that either a path of eccentricity $3k$ has been discovered, or one of the explored possibilities corresponds to eight extensions of size at least k starting from $[i_{min}^{s,l}, i_{max}^{s,l}]$.

In the latter case, Equation (14) implies that the final couple of vertices (x, y) fulfills $i_{min}^{x,y} \leq k$ and $i_{max}^{x,y} \geq t - k$. Every vertex of P is then of distance at most $2k$ of $Q_{x,y}$ and thus $Q_{x,y}$ is of eccentricity at most $3k$.

Proof (Complexity). The algorithm computes two BFS trees at line 1 and 2, taking $\mathcal{O}(n + m)$ time. The rest of the operations is computed in constant time.

The recursivity width is 2 and, since it is first called with $step = 0$, the recursivity length is 8. The algorithm is thus called 255 times. Therefore the total runtime of the algorithm is $\mathcal{O}(n + m)$.

Proof (Tightness of the approximation). Figure 3 shows a graph for which the algorithm may produce a path of eccentricity $3k(G)$ (see caption).
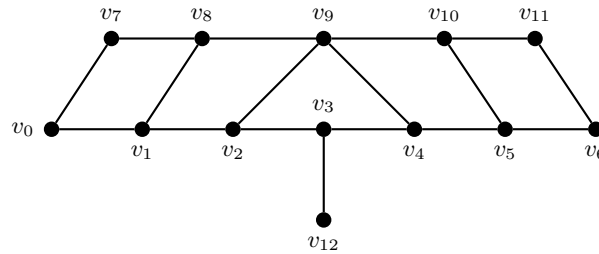

Fig. 3. Tightness of the bound shown in Theorem 3. The algorithm may indeed loop between the following couples of vertices : $(v_0, v_6), (v_0, v_{12}), (v_6, v_{12}), (v_0, v_{11}), (v_{11}, v_{12}), (v_6, v_7), (v_7, v_{12}), (v_{11}, v_7)$. Each time, it may choose a shortest path of eccentricity 3 (passing through v_8 v_9 and v_{10} whenever v_{12} is not an endvertex of the path) while $v_0..v_3..v_6$ has eccentricity 1.

4 Bounds between MESP and Laminarity

In this section, we investigate the link between the MESP problem and the notion of laminarity introduced by Völkel *et al.* in [12]. The study of the k-laminar graph class finds motivation both from a theoretical and practical point of view. On the theoretical side, AT-free graphs form a well known graph class introduced half a century ago by Lekkerkerker and Boland [10], which contains many graph classes like co-comparability graphs. An AT-free graph admits a diameter all other vertices are adjacent with [5]. It is then natural to extend this notion of dominating diameter. On the practical side, some large graphs constructed from reads similarity networks of genomic or metagenomic data appear to have a very long diameter and all vertices at short distance from it [12], and exhibiting the "best" diameter allows to better understand their structure.

Definition 2 (laminarity). *A graph G is*

- *l-laminar if G has a diameter of eccentricity at most l.*
- *s-strongly laminar if every diameter has eccentricity at most s.*

$l(G)$ and $s(G)$ denote the minimal values of l and s such that G is respectively l-laminar and s-strongly laminar.

A natural question about laminarity and MESP is to ask what link exists between them.

Theorem 4. *For every graph G,*

$$k(G) \leq l(G) \leq 4k(G) - 2$$
$$k(G) \leq s(G) \leq 4k(G)$$

Moreover, there exist three graph sequences $(G_k)_{k \geq 1}$, $(H_k)_{k \geq 1}$ and $(J_k)_{k \geq 1}$ such that, for every k,

- *$k(G_k) = l(G_k) = s(G_k) = k$;*
- *$k(H_k) = k$ and $l(H_k) = 4k - 2$;*
- *$k(J_k) = k$ and $s(J_k) = 4k$;*

The bounds given by the inequalities are therefore tight.

Proof ($k(G) \leq l(G)$ and $k(G) \leq s(G)$).
Those inequalities are straightforward as every diameter is by definition a shortest path. The eccentricity of every diameter is therefore always greater than $k(G)$.

Proof ($s(G) \leq 4k(G)$).
Let $D = x_0, x_1, ... x_s$ be a diameter of G and $P = v_0, v_1 ... v_t$ a shortest path of eccentricity k. We shall show $ecc(D) \leq 4k$. Let z be any vertex of G. Since $ecc(P) = k$ there exists a vertex v_i of P such that $d(z, v_i) \leq k$. Let us distinguish

three cases:

- Case 1: there exists vertices x_a, x_b of D and v_a, v_b of P such that $a \le i \le b$ and $d(v_a, x_a) \le k$ and $d(v_b, x_b) \le k$. Then by Lemma 1, z is at distance at most $3k$ from any shortest path between x_a and x_b, and thus at distance at most $3k$ of D.

- Case 2: there exists no vertex v_a of P with $a \le i$ and $d(v_a, D) \le k$
- Case 3: there exists no vertex v_a of P with $i \le a$ and $d(v_a, D) \le k$.

Without loss of generality we focus on Case 2 (illustrated in Figure 4), which is symmetric with Case 3. Let l (resp. m) be such that v_l (resp. v_m) is at distance at most k of x_0 (resp. x_s), assume $l \le m$:

$$d(v_l, v_m) \ge d(x_0, x_s) - 2k \tag{15}$$

D being a diameter,

$$d(x_0, x_s) \ge d(v_0, v_t) \tag{16}$$

By combining those inequalities,

$$d(v_l, v_m) \ge d(v_0, v_t) - 2k \tag{17}$$

$$d(v_l, v_m) \ge d(v_0, v_i) + d(v_i, v_l) + d(v_l, v_m) + d(v_m, v_t) - 2k \tag{18}$$

$$2k \ge d(v_i, v_l) \tag{19}$$

It follows that z is at distance at most $4k$ of x_0.

Proof $(l(G) \le 4k(G) - 2)$.

Let $D = x_0, x_1, ...x_s$ be a diameter of G and $P = v_0, v_1...v_t$ a shortest path of eccentricity k. We shall show that either $ecc(D) \le 4k - 2$ or G contains a diameter D' of eccentricity $3k$. If P is a diameter we are done. Let us suppose from now it is of length at most $|D| - 1$.

Let z be any vertex of G and v_i a vertex of P such that $d(z, v_i) \le k$. Let us distinguish the same three cases than in the proof that $s(G) \le 4k(G)$. The first case also leads to $d(z, D) \le 3k$. The second and third being symmetric, let us suppose there exists no vertex v_j of P at distance at most k of D such that $j \le i$.

Let v_l (resp. v_m) be a vertex of P at distance at most k from x_0 (resp. x_s), clearly,

$$d(v_l, v_m) \ge |D| - 2k. \tag{20}$$

Let us distinguish two subcases:

- Case 2.1: $d(v_l, v_m) > |D| - 2k$,

$$d(v_i, v_l) \le d(v_0, v_t) - d(v_l, v_m) \le (|D| - 1) - (|D| - 2k + 1) \le 2k - 2 \tag{21}$$

It follows that z is at distance at most $4k - 2$ of D.

• Case 2.2: $d(v_l, v_m) = |D| - 2k$

In this case, a path $D' = x_0, ..v_l, v_{l+1}, ..v_m, ..x_s$ is a diameter. Assuming $l \leq m$, Equation 19 in previous proof shows that:

$$d(v_i, v_l) \leq 2k \qquad (22)$$

and with a symmetrical reasoning,

$$d(v_m, v_t) \leq 2k \qquad (23)$$

It follows that any vertex v of G at distance at most k of a vertex v_a with $a \leq l$ (resp. $a \geq m$) is at distance at most $3k$ of v_l (resp. v_m). Hence at distance at most $3k$ of D'. $v_l, v_{l+1}, ..v_m$ being a subpath of D', any vertex v of G at distance at most k of a vertex v_a with a between m and t is at distance at most k of D'. Finally, any vertex of G is at distance at most $3k$ of D'.
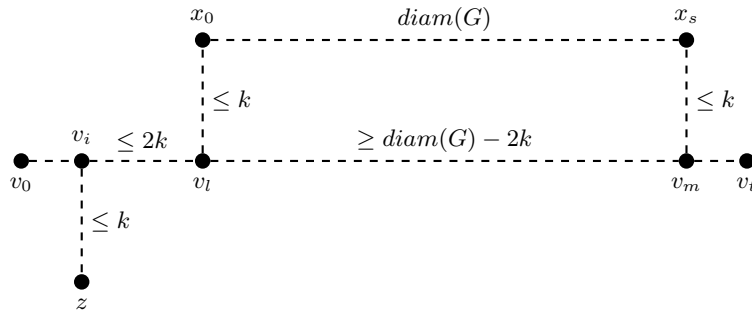

Fig. 4. Notations used in Case 2 of the proof of Theorem 4

Proof (Tightness of the bounds).

Consider the graph G_k reduced to a path P of length $4k$ to which a second path of length k is attached in the middle. P is then simultaneously the only diameter and the MESP, and it k-covers G_k but doesn't $(k-1)$-cover it. Hence the inequalities $k(G) \leq l(G)$ and $k(G) \leq s(G)$ are tight.

Figure 5 shows how to build the graph sequence $(J_k)_{k \geq 1}$ (only J_1 and J_6 are drawn). J_k is a graph with a shortest path of eccentricity k and a diameter of eccenticity $4k$. The inequality $s(G) \leq 4k(G)$ is thus tight.

Figure 6 shows how to build the graph sequence $(H_k)_{k \geq 1}$ (only H_1, H_2 and and H_6 are drawn). H_k is a graph with a shortest path of eccentricity k, while the unique diameter has eccenticity $4k - 2$ (H_1 is a special case with two diameters). The inequality $l(G) \leq 4k(G) - 2$ is therefore tight.
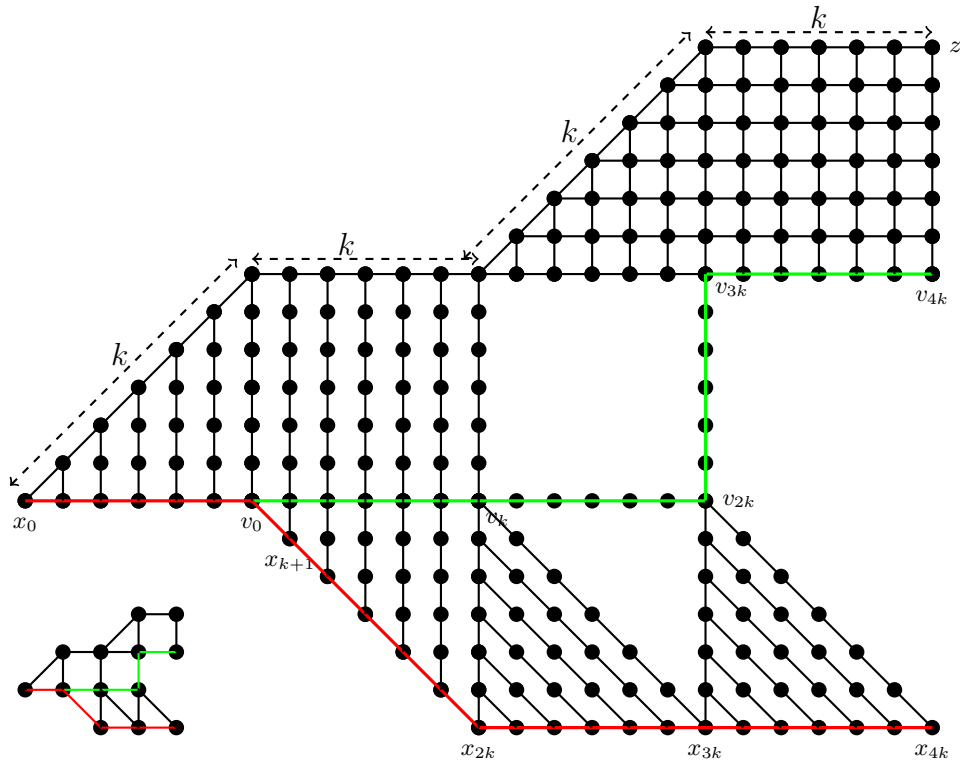
Fig. 5. Proof that $s(G) \geq 4k(G)$. The red path $x_0, x_1, ... x_{4k}$ is a diameter of length $4k$ and at distance $4k$ of z; while the green path $v_0, v_1, ... v_{4k}$ is a shortest path (another diameter indeed) of eccentricity k. The large graph is J_6 (using the graph sequence $(J_k)_k$ from Theorem 4) and the small one on the bottom left is J_1. The other members of the sequence car easily be derived.

.

5 Conclusion

We have investigated the Minimum Eccentricity Shortest Path problem for general graphs and proposed a linear time algorithm computing a 3-approximation. The algorithm is a 2-recursive function with constant recursivity depth, launching two BFSs each time, thus taking linear time. Additionally, we've established some tight bounds linking the MESP parameter $k(G)$ and the k-laminarity parameters $s(G)$ and $l(G)$.

On improving the current approximation algorithms, the following remark should be noted. Our algorithm is confined in finding a good pair of vertices in the graph, and the shortest path between them is then picked arbitrarily. By doing so, we are unlikely to get a better result than a 3-approximation. Indeed as shown by [9] there exist graphs for which the MESP solution is a path of

eccentricity k between two vertices s and t such that some other shortest paths between s and t have an eccentricity of exactly $3k$.

About laminarity parameters, computing $l(G)$ is NP-complete, while computing $s(G)$ can be done in $O(n^2 m \log n)$ time [12]. It may be interesting to design an approximation algorithm, *i.e* producing a diameter of eccentricity at most $\alpha s(G)$ or $\beta l(G)$. Linear-time algorithms like BFS cannot be used however, since we do not know how to compute $diam(G)$ faster than a matrix product,
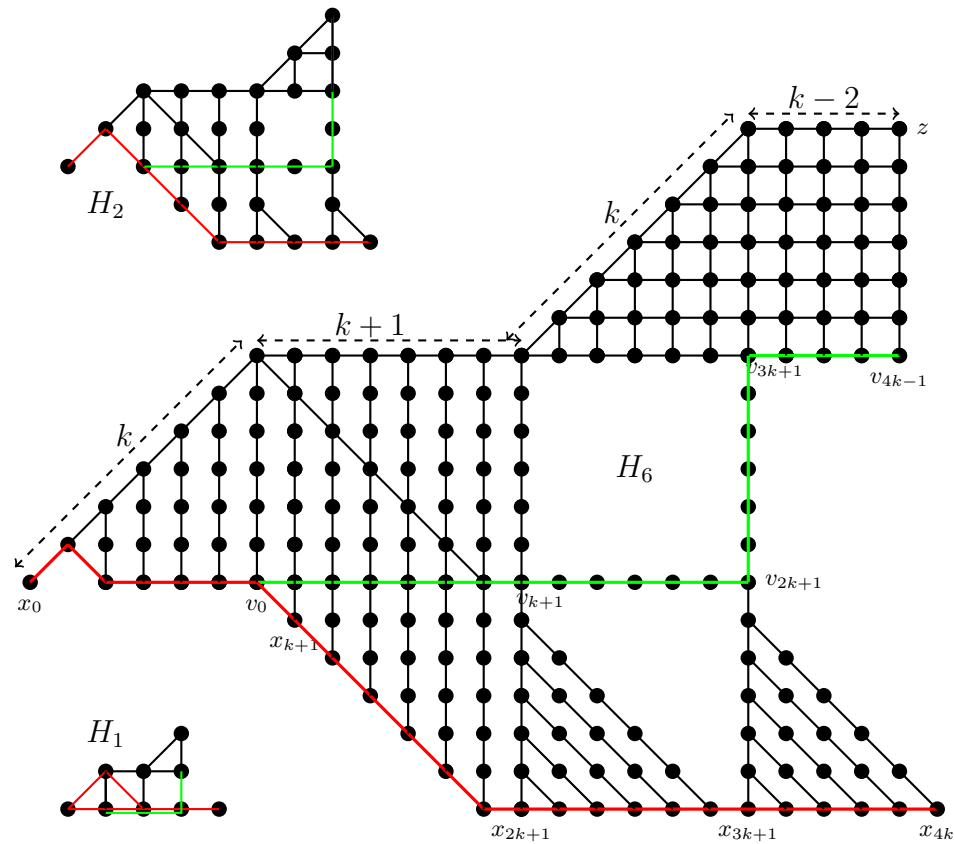

Fig. 6. Proof that $l(G) \geq 4k(G) - 2$. It is a graph sequence $(H_k)_k$, using the notation from Theorem 4. For $k \geq 2$, the red path $x_0, x_1, ... x_{4k}$ is the unique diameter. Its length is $4k$ and it is at distance $4k - 2$ of z. The green path $v_0, v_1, ... v_{4k-1}$ is a shortest path of length $4k - 1$ and of eccentricity k. Graphs H_2 and H_6 are drawn but all graphs H_k, $k \geq 2$ can be derived from the pattern of H_6. The small graph on the bottom left is the special case H_1 who do not follow this pattern. It admits exactly two diameters, both of eccentricity 2 (red), and a shortest path of eccentricity 1 (green).

and even surlinear approximation are studied [1]. Different techniques than the ones used here must therefore be employed.

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